ON THE ERDÖS-FALCONER DISTANCE PROBLEM FOR TWO SETS OF DIFFERENT SIZE IN VECTOR SPACES OVER FINITE FIELDS

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ABSTRACT. We consider a finite fields version of the Erdös-Falconer distance problem for two different sets. In a certain range for the sizes of the two sets we obtain results of the conjectured order of magnitude.

1. INTRODUCTION

Let $E \subset \mathbb{R}^s$, and let

$$\Delta(E) = \{||\mathbf{x}-\mathbf{y}|| : \mathbf{x}, \mathbf{y} \in E\}$$

be the set of distances between elements in E, where $|| \cdot ||$ denotes the Euclidean metric. Erdös' distance conjecture [2] is that

$$\#\Delta(E) \gg_\epsilon (\#E)^{s/2-\epsilon}$$

for $s \geq 2$ and finite E. In a recent breakthrough paper by Guth and Katz [4], this problem has been solved for $s = 2$, whereas it is still open for higher dimensions. Later Falconer [3] considered a continuous version of Erdös' distance problem, replacing $\#E$ by the Hausdorff dimension of E, and $\#\Delta(E)$ by the Lebesgue measure of $\Delta(E)$. More recently, Iosevich and Rudnev [5] dealt with a finite fields version of these problems. For a finite field $\mathbb{F}_q$ and $\mathbf{x} \in \mathbb{F}_q^s$, let

$$|\mathbf{x}|^2 = \sum_{i=1}^{s} x_i^2.$$

In the following we will always assume that q is odd; in particular, $q \geq 3$. Then one of Iosevich and Rudnev's main results is that if $E \subset \mathbb{F}_q^s$ where $\#E \geq Cq^{s/2}$ for a sufficiently large absolute constant C, then

$$\#\Delta(E) \gg \min\left\{q, \frac{\#E}{q^{(s-1)/2}}\right\}, \tag{1}$$

where

$$\Delta(E) = \left\{|\mathbf{x}-\mathbf{y}|^2 : \mathbf{x}, \mathbf{y} \in E\right\}.$$

In particular, if $\#E \gg q^{(s+1)/2}$, then $\#\Delta(E) \gg q$. For $s = 2$, the stronger bound

$$\#\Delta(E) \gg \min\left\{q, \frac{(\#E)^{3/2}}{q}\right\},$$

has recently been established by Chapman, Erdogan, Hart, Iosevich and Koh (see [1]). This bound is stronger than (1) for $\#E \gg q$. Our focus in this paper is on

2000 *Mathematics Subject Classification.* 11T24, 52C10.

a generalisation of this problem to the situation of distances between two different sets $E, F \in \mathbb{F}_q^s$. Analogously to above, we define

$$\Delta(E,F) = \#\{|\mathbf{x}-\mathbf{y}|^2 : \mathbf{x} \in E,\, \mathbf{y} \in F\}.$$

It is straightforward to adapt Iosevich and Rudnev's approach to show that if $(\#E)(\#F) \geq Cq^s$ for a sufficiently large constant C, then

$$\#\Delta(E,F) \gg \min\left\{q, \frac{(\#E)^{1/2}(\#F)^{1/2}}{q^{(s-1)/2}}\right\}. \tag{2}$$

In particular, if $(\#E)(\#F) \gg q^{s+1}$, then $\#\Delta(E,F) \gg q$. For $s = 2$, the stronger result that $\#\Delta(E,F) \gg q$ if

$$(\#E)(\#F) \gg q^{8/3} \tag{3}$$

has recently been proved by Koh and Shen ([6], Theorem 1.3), and they also put forward the following conjecture (see Conjecture 1.2 in [7]) generalising Conjecture 1.1 in [5] for even s.

Conjecture 1. *Let $s \geq 2$ be even and $(\#E)(\#F) \geq Cq^s$ for a sufficiently large absolute constant C. Then $\#\Delta(E,F) \gg q$.*

In this paper we establish the following result, which improves on (2) and (3) for sets E, F of different size in a certain range for $(\#E)$ and $(\#F)$.

Theorem 1. *Let $E, F \subset \mathbb{F}_q^s$ where $s \geq 2$. Further, let $\#E \leq \#F$ and $(\#E)(\#F) \geq 900q^s$. Then*

$$\#\Delta(E,F) \gg \min\left\{q, \frac{\#F}{q^{(s-1)/2}\log q}\right\}. \tag{4}$$

For $s = 2$ also the alternative lower bound

$$\#\Delta(E,F) \gg \min\left\{q, \frac{(\#E)^{1/2}\#F}{q\log q}\right\} \tag{5}$$

holds true.

Note that (5) is superior to (4) for $s = 2$ if and only if $\#E \gg q$. Note also that Theorem 1 implies that if $(\#E)(\#F) \geq 900q^s$ and $\#F \geq q^{(s+1)/2}\log q$, then $\#\Delta(E,F) \gg q$. These conditions on E and F are for example satisfied if $\#E \geq 900q^{(s-1)/2}$ and $\#F \geq q^{(s+1)/2}\log q$. Hence apart from a factor $\log q$, Conjecture 1 holds true for a certain range of cardinalities of E and F, both for even and odd dimension s.

Our approach follows that of Iosevich and Rudnev, paying close attention to certain spherical averages of Fourier transforms.

2. Notation

Our notation is fairly standard. Let $\mathbb{C}$ be the field of complex numbers, and we write $\mathbb{F}_q$ for a fixed finite field having q elements, where q is odd, and we denote by $\mathbb{F}_q^*$ the non-zero elements of $\mathbb{F}_q$. Further, if $a \in \mathbb{F}_q^*$, we write $\overline{a}$ for the multiplicative inverse of a. Moreover, we write

$$e\left(\frac{j}{q}\right) \ (1 \leq j \leq q)$$

for the additive characters of $\mathbb{F}_q$, the main character being that where $j = q$. If q is a prime, then $e(j/q)$ is just

$$e\left(\frac{j}{q}\right) = e^{2\pi i \frac{j}{q}}$$

where $i^2 = -1$. If $f : \mathbb{F}_q^s \to \mathbb{C}$ is any function, then we denote by $\hat{f}$ its Fourier transform given by

$$\hat{f}(\mathbf{x}) = q^{-s} \sum_{\mathbf{m}\in\mathbb{F}_q^s} e\left(\frac{-\mathbf{mx}}{q}\right) f(\mathbf{m}),$$

where as usual $\mathbf{mx}$ is the inner product

$$\mathbf{mx} = \sum_{i=1}^{s} m_i x_i.$$

The function f can be reconstructed from its Fourier transform $\hat{f}$ via the inversion formula

$$f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{F}_q^s} e\left(\frac{\mathbf{mx}}{q}\right) \hat{f}(\mathbf{m}).$$

The tool that is most important for us is *Plancherel's formula*

$$\sum_{\mathbf{m}\in\mathbb{F}_q^s} \left|\hat{f}(\mathbf{m})\right|^2 = q^{-s} \sum_{\mathbf{x}\in\mathbb{F}_q^s} |f(\mathbf{x})|^2 .$$

All these formulas are easy to verify, and proofs can be found in many textbooks on number theory or Fourier analysis. For a subset $E \subset \mathbb{F}_q^s$, we also write E for its characteristic function, i.e.

$$E(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in E, \\ 0 & \text{otherwise,} \end{cases}$$

and analogously for subsets $F \subset \mathbb{F}_q^s$. Moreover, let S_r be the sphere

$$S_r = \{\mathbf{x} \in \mathbb{F}_q^s : |\mathbf{x}|^2 = r\},$$

and as above we also write S_r for the corresponding characteristic function. Moreover, for $E \subset \mathbb{F}_q^s$ and $r \in \mathbb{F}_q$, let $\sigma_E(r)$ be the spherical average

$$\sigma_E(r) = \sum_{\mathbf{a}\in\mathbb{F}_q^s : |\mathbf{a}|^2 = r} |\hat{E}(\mathbf{a})|^2$$

of the Fourier transform $\hat{E}(\mathbf{a})$ of E, and we define analogously $\sigma_F(r)$. Furthermore, we define

$$\sigma_{E,F}(r) = \sum_{\mathbf{m}\in\mathbb{F}_q^s : |\mathbf{m}|^2 = r} \overline{\hat{E}(\mathbf{m})} \hat{F}(\mathbf{m}),$$

where as usual $^{-}$ denotes complex conjugation. In particular, $\sigma_E(r) = \sigma_{E,E}(r)$. Our main tool for bounding $\#\Delta(E, F)$ below is the following upper bound on $\sigma_E \sigma_F$ on average.

Lemma 1. *Notation as above. Then we have*

$$\sum_{r\in\mathbb{F}_q^*} \sigma_E(r)\sigma_F(r) \ll \log q \left(q^{-2s-1}(\#E)(\#F) + q^{-\frac{5s+1}{2}}(\#E)^2(\#F)\right). \tag{6}$$

For $s=2$ also the alternative bound

$$\sum_{r\in\mathbb{F}_q^*}\sigma_E(r)\sigma_F(r)\ll(\log q)q^{-5}(\#E)^{3/2}(\#F) \tag{7}$$

holds true.

Note that (7) is superior to (6) for $s=2$ if and only if $\#E\gg q$. Finally, for fixed $E,F\in\mathbb{F}_q^s$ and given $j\in\mathbb{F}_q$ we define

$$\nu(j)=\#\{(\mathbf{x},\mathbf{y})\in E\times F:|\mathbf{x}-\mathbf{y}|^2=j\}. \tag{8}$$

3. Proof of Lemma 1

Clearly, $|\hat{F}(\mathbf{a})|\le q^{-s}(\#F)$, thus

$$\sigma_F(r)=\sum_{\mathbf{a}\in\mathbb{F}_q^s:|\mathbf{a}|^2=r}\left|\hat{F}(\mathbf{a})\right|^2\le q^{-s}(\#F)^2\le q^s.$$

Hence, by a dyadic intersection of the range of possible values of σ_F we can find a subset $M\subset\mathbb{F}_q^*$ such that

$$\sum_{r\in\mathbb{F}_q^*}\sigma_E(r)\sigma_F(r)\ll\log q\sum_{r\in M}\sigma_E(r)\sigma_F(r) \tag{9}$$

and

$$A\le\sigma_F(r)\le 2A \tag{10}$$

for all $r\in M$, for a suitable positive constant A. By Cauchy-Schwarz,

$$\sum_{r\in M}\sigma_E(r)\sigma_F(r)\le\left(\sum_{r\in M}\sigma_E(r)^2\right)^{1/2}\left(\sum_{r\in M}\sigma_F(r)^2\right)^{1/2}. \tag{11}$$

Let us first bound $\sum_{r\in M}\sigma_E(r)^2$. To this end, we need the following result.

Lemma 2. *Let $r\in\mathbb{F}_q^*$. Then*

$$\sigma_E(r)\ll q^{-s-1}\#E+q^{-\frac{3s+1}{2}}(\#E)^2. \tag{12}$$

For $s=2$, we also have the alternative bound

$$\sigma_E(r)\ll q^{-3}(\#E)^{3/2}. \tag{13}$$

Proof. For (12), see the proof of Lemma 1.8 in [5]. Note that the first term on the right hand side is missing in the statement of Lemma 1.8 in [5], but it shows up in the proof of the Lemma, and is clearly needed as for example shown by choosing $E=\{\mathbf{0}\}$. The second bound (13) is Lemma 4.4 in [1]. □

Using Lemma 2, we obtain

$$\sum_{r\in M}\sigma_E(r)^2\le\left(\max_{t\in\mathbb{F}_q^*}\sigma_E(t)\right)^2\#M\ll(\#M)\left(q^{-2s-2}(\#E)^2+q^{-3s-1}(\#E)^4\right) \tag{14}$$

in general, and for $s=2$ we also obtain the alternative bound

$$\sum_{r\in M}\sigma_E(r)^2\ll(\#M)q^{-6}(\#E)^3. \tag{15}$$

Next, let us bound $\sum_{r\in M}\sigma_F(r)^2$.

Lemma 3. *We have*
$$\sum_{r\in\mathbb{F}_q}\sigma_F(r)=q^{-s}\#F.$$

Proof. Since
$$\sum_{r\in\mathbb{F}_q}\sigma_F(r)=\sum_{\mathbf{a}\in\mathbb{F}_q^s}|\hat{F}(\mathbf{a})|^2,$$
the result follows immediately from Plancherel's formula
$$\sum_{\mathbf{a}\in\mathbb{F}_q^s}|\hat{F}(\mathbf{a})|^2=q^{-s}\sum_{\mathbf{a}\in\mathbb{F}_q^s}F(a)^2=q^{-s}\#F.$$
□

We start with the observation that by (10), we have
$$\sum_{r\in M}\sigma_F(r)^2\le 4\cdot\#M\cdot A^2. \tag{16}$$

Next, by Lemma 3,
$$q^{-2s}(\#F)^2=\left(\sum_{r\in\mathbb{F}_q}\sigma_F(r)\right)^2=\sum_{m,n\in\mathbb{F}_q}\sigma_F(m)\sigma_F(n). \tag{17}$$

Moreover, by (10),
$$\sum_{m,n\in\mathbb{F}_q}\sigma_F(m)\sigma_F(n)\ge\sum_{m,n\in M}\sigma_F(m)\sigma_F(n)\gg(\#M)^2A^2. \tag{18}$$

By (16), (17), and (18) we obtain
$$\begin{aligned}\sum_{r\in M}\sigma_F(r)^2 &\ll \#M\cdot A^2\ll(\#M)^{-1}\sum_{m,n\in M}\sigma_F(m)\sigma_F(n)\\ &\ll (\#M)^{-1}q^{-2s}(\#F)^2. \end{aligned}\tag{19}$$

Summarising (9), (11), (14) and (19), we obtain
$$\sum_{r\in\mathbb{F}_q^*}\sigma_E(r)\sigma_F(r)\ll(\log q)\left(q^{-2s-1}(\#E)(\#F)+q^{-\frac{5s+1}{2}}(\#E)^2(\#F)\right).$$

Using (15) instead of (14), for $s=2$ we also obtain
$$\sum_{r\in\mathbb{F}_q^*}\sigma_E(r)\sigma_F(r)\ll(\log q)q^{-5}(\#E)^{3/2}(\#F).$$

This completes the proof of Lemma 1.

4. Preparations for the proof of Theorem 1

Before we are able to prove Theorem 1, we first need to collect some useful lemmas.

Lemma 4. *For* $\mathbf{m}\in\mathbb{F}_q^s$*, let*
$$\chi(\mathbf{m})=\begin{cases}1 & \textit{if } \mathbf{m}=\mathbf{0}\\ 0 & \textit{if } \mathbf{m}\ne\mathbf{0}.\end{cases}$$

Then

$$\hat{S}_r(\mathbf{m}) = \frac{\chi(\mathbf{m})}{q} + q^{-\frac{s}{2}-1} c_q^s \sum_{j \in \mathbb{F}_q^*} e\left(\frac{jr + |\mathbf{m}|^2 \overline{4j}}{q}\right),$$

where the complex number c_q depends only on q and s, and $|c_q| = 1$.

Proof. See formula (2.12) in [5]. □

Lemma 5. *Let $j \in \mathbb{F}_q$. Then*

$$\nu(j) = \frac{(\#E)(\#F)}{q} + \delta(j) + \epsilon(j)$$

where

$$\delta(j) = q^{2s} \sum_{\mathbf{m} \in \mathbb{F}_q^s : \mathbf{m} \neq \mathbf{0}} \hat{S}_j(\mathbf{m}) \overline{\hat{E}(\mathbf{m})} \hat{F}(\mathbf{m}) \tag{20}$$

and

$$|\epsilon(j)| \leq (\#E)(\#F) q^{-1}.$$

Proof. We have

$$\begin{aligned}
\nu(j) &= \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^s} E(\mathbf{x}) F(\mathbf{y}) S_j(\mathbf{x} - \mathbf{y}) \\
&= \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^s} E(\mathbf{x}) F(\mathbf{y}) \sum_{\mathbf{m} \in \mathbb{F}_q^s} e\left(\frac{(\mathbf{x} - \mathbf{y})\mathbf{m}}{q}\right) \hat{S}_j(\mathbf{m}) \\
&= \sum_{\mathbf{m} \in \mathbb{F}_q^s} \hat{S}_j(\mathbf{m}) \left(\sum_{\mathbf{x} \in \mathbb{F}_q^s} E(\mathbf{x}) e\left(\frac{\mathbf{x}\mathbf{m}}{q}\right)\right) \left(\sum_{\mathbf{y} \in \mathbb{F}_q^s} F(\mathbf{y}) e\left(\frac{-\mathbf{y}\mathbf{m}}{q}\right)\right) \\
&= q^{2s} \sum_{\mathbf{m} \in \mathbb{F}_q^s} \hat{S}_j(\mathbf{m}) \overline{\hat{E}(\mathbf{m})} \hat{F}(\mathbf{m}).
\end{aligned}$$

Now

$$\overline{\hat{E}(\mathbf{0})} = q^{-s} \#E$$

and

$$\hat{F}(\mathbf{0}) = q^{-s} \#F.$$

The result now follows immediately from Lemma 4. □

Lemma 6. *Let $(\#E)(\#F) \geq 900 q^s$. Then*

$$\nu(0) \leq \frac{21}{30} (\#E)(\#F).$$

Proof. By Lemma 5, we have

$$\nu(0) = \frac{(\#E)(\#F)}{q} + \delta(0) + \epsilon(0)$$

where

$$\delta(0) = q^{2s} \sum_{\mathbf{m} \in \mathbb{F}_q^s : \mathbf{m} \neq \mathbf{0}} \hat{S}_0(\mathbf{m}) \overline{\hat{E}(\mathbf{m})} \hat{F}(\mathbf{m})$$

and

$$|\epsilon(0)| \leq \frac{(\#E)(\#F)}{q}.$$

Now Lemma 4 yields

$$\left|\hat{S}_0(\mathbf{m})\right| \leq q^{-s/2}$$

for $\mathbf{m} \neq \mathbf{0}$. Hence, by Cauchy-Schwarz and Plancherel's formula,

$$\begin{aligned} |\delta(0)| &\leq q^{\frac{3}{2}s}\left(\sum_{\mathbf{m}\in\mathbb{F}_q^s}|\hat{E}(\mathbf{m})|^2\right)^{1/2}\left(\sum_{\mathbf{m}\in\mathbb{F}_q^s}|\hat{F}(\mathbf{m})|^2\right)^{1/2} \\ &\leq q^{s/2}(\#E)^{1/2}(\#F)^{1/2}. \end{aligned}$$

Since $(\#E)(\#F) \geq 900q^s$, we conclude that

$$|\delta(0)| \leq \frac{(\#E)(\#F)}{30}.$$

Therefore, since $q \geq 3$, we have

$$\nu(0) \leq 2\frac{(\#E)(\#F)}{q} + |\delta(0)| \leq \frac{21}{30}(\#E)(\#F).$$

□

Lemma 7. *Let $\delta(j)$ be defined in (20). Then*

$$\sum_{j\in\mathbb{F}_q}|\delta(j)|^2 \leq q^{3s}\left|\sigma_{E,F}(0)\right|^2 + q^{3s}\sum_{r\in\mathbb{F}_q^*}\sigma_E(r)\sigma_F(r) + q^{s-1}(\#E)(\#F).$$

Proof. By (20), we have

$$\sum_{j\in\mathbb{F}_q}|\delta(j)|^2 = q^{4s}\sum_{j\in\mathbb{F}_q}\sum_{\mathbf{m},\mathbf{n}\in\mathbb{F}_q^s:\mathbf{m},\mathbf{n}\neq\mathbf{0}}\hat{S}_j(\mathbf{m})\overline{S_j(\mathbf{n})}\overline{\hat{E}(\mathbf{m})}\hat{F}(\mathbf{m})\hat{E}(\mathbf{n})\overline{\hat{F}(\mathbf{n})}.$$

Using Lemma 4, we obtain

$$\sum_{j\in\mathbb{F}_q}|\delta(j)|^2 = q^{3s-2}\sum_{\mathbf{m},\mathbf{n}\in\mathbb{F}_q^s:\mathbf{m},\mathbf{n}\neq\mathbf{0}}\overline{\hat{E}(\mathbf{m})}\hat{F}(\mathbf{m})\hat{E}(\mathbf{n})\overline{\hat{F}(\mathbf{n})}T(\mathbf{m},\mathbf{n}),$$

where

$$\begin{aligned} T(\mathbf{m},\mathbf{n}) &= c_q^s\overline{c}_q^s\sum_{j\in\mathbb{F}_q}\sum_{k\in\mathbb{F}_q^*}e\left(\frac{kj+|\mathbf{m}|^2\bar{4}\bar{k}}{q}\right)\sum_{l\in\mathbb{F}_q^*}e\left(\frac{-lj-|\mathbf{n}|^2\bar{4}\bar{l}}{q}\right) \\ &= q\sum_{k\in\mathbb{F}_q^*}e\left(\frac{\bar{4}\bar{k}(|\mathbf{m}|^2-|\mathbf{n}|^2)}{q}\right) \\ &= q\left(\sum_{k\in\mathbb{F}_q}e\left(\frac{\bar{4}k(|\mathbf{m}|^2-|\mathbf{n}|^2)}{q}\right)-1\right) \\ &= \begin{cases} q^2-q & \text{if } |\mathbf{m}|^2=|\mathbf{n}|^2 \\ -q & \text{if } |\mathbf{m}|^2\neq|\mathbf{n}|^2. \end{cases} \end{aligned}$$

Hence

$$\sum_{j\in\mathbb{F}_q}|\delta(j)|^2 \leq U+|V| \tag{21}$$

where

$$U = q^{3s}\sum_{\mathbf{m},\mathbf{n}\in\mathbb{F}_q^s:|\mathbf{m}|^2=|\mathbf{n}|^2}\overline{\hat{E}(\mathbf{m})}\hat{F}(\mathbf{m})\hat{E}(\mathbf{n})\overline{\hat{F}(\mathbf{n})} = q^{3s}\sum_{r\in\mathbb{F}_q}\left|\sigma_{E,F}(r)\right|^2.$$

and

$$V = q^{3s-1} \sum_{\mathbf{m},\mathbf{n} \in \mathbb{F}_q^s} \overline{\hat{E}(\mathbf{m})} \hat{F}(\mathbf{m}) \hat{E}(\mathbf{n}) \overline{\hat{F}(\mathbf{n})}.$$

By Cauchy-Schwarz' inequality,

$$|\sigma_{E,F}(r)|^2 \leq \left(\sum_{\mathbf{m} \in \mathbb{F}_q^s : |\mathbf{m}|^2 = r} |\hat{E}(\mathbf{m})|^2 \right) \left(\sum_{\mathbf{m} \in \mathbb{F}_q^s : |\mathbf{m}|^2 = r} |\hat{F}(\mathbf{m})|^2 \right) = \sigma_E(r) \sigma_F(r).$$

Thus

$$U \leq q^{3s} |\sigma_{E,F}(0)|^2 + q^{3s} \sum_{r \in \mathbb{F}_q^*} \sigma_E(r) \sigma_F(r). \tag{22}$$

Another application of Cauchy-Schwarz shows that

$$\begin{aligned} \left| \sum_{\mathbf{m},\mathbf{n} \in \mathbb{F}_q^s} \overline{\hat{E}(\mathbf{m})} \hat{F}(\mathbf{m}) \hat{E}(\mathbf{n}) \overline{\hat{F}(\mathbf{n})} \right| &\leq \left(\sum_{\mathbf{m} \in \mathbb{F}_q^s} \left| \hat{E}(\mathbf{m}) \right| \left| \hat{F}(\mathbf{m}) \right| \right)^2 \\ &\leq \sum_{\mathbf{m} \in \mathbb{F}_q^s} \left| \hat{E}(\mathbf{m}) \right|^2 \sum_{\mathbf{m} \in \mathbb{F}_q^s} \left| \hat{F}(\mathbf{m}) \right|^2 . \end{aligned}$$

Hence, by Plancherel's formula,

$$|V| \leq q^{s-1} (\#E)(\#F). \tag{23}$$

The result now follows from (21), (22) and (23). □

Lemma 8. *Let $s \geq 2$, $(\#E) \leq (\#F)$ and $(\#E)(\#F) \geq 900 q^s$. Then we have*

$$|\sigma_{E,F}(0)|^2 = q^{-3s} \nu(0)^2 + O\left(q^{-3s-1} (\#E)^2 (\#F)^2\right).$$

Proof. We have

$$\begin{aligned} \sigma_{E,F}(0) &= \sum_{\mathbf{m} \in \mathbb{F}_q^s : |\mathbf{m}|^2 = 0} \overline{\hat{E}(\mathbf{m})} \hat{F}(\mathbf{m}) = \sum_{\mathbf{m} \in \mathbb{F}_q^s} \overline{\hat{E}(\mathbf{m})} \hat{F}(\mathbf{m}) S_0(\mathbf{m}) \\ &= q^{-2s} \sum_{\mathbf{m} \in \mathbb{F}_q^s} \sum_{\mathbf{x} \in \mathbb{F}_q^s} E(\mathbf{x}) e\left(\frac{\mathbf{m}\mathbf{x}}{q} \right) \sum_{\mathbf{y} \in \mathbb{F}_q^s} F(\mathbf{y}) e\left(\frac{-\mathbf{m}\mathbf{y}}{q} \right) S_0(\mathbf{m}) \\ &= q^{-2s} \sum_{\mathbf{x},\mathbf{y} \in \mathbb{F}_q^s} E(\mathbf{x}) F(\mathbf{y}) \sum_{\mathbf{m} \in \mathbb{F}_q^s} e\left(\frac{\mathbf{m}(\mathbf{x}-\mathbf{y})}{q} \right) S_0(\mathbf{m}) \\ &= q^{-s} \sum_{\mathbf{x},\mathbf{y} \in \mathbb{F}_q^s} E(\mathbf{x}) F(\mathbf{y}) \hat{S}_0(\mathbf{y}-\mathbf{x}). \end{aligned}$$

By Lemma 4 and Cauchy-Schwarz' inequality we obtain

$$\begin{aligned}
\sigma_{E,F}(0) &= q^{-s}c_q^s \sum_{\mathbf{x},\mathbf{y}\in\mathbb{F}_q^s:\mathbf{x}\neq\mathbf{y},|\mathbf{x}-\mathbf{y}|^2=0} E(\mathbf{x})F(\mathbf{y})\left(q^{-s/2}-q^{-s/2-1}\right)\\
&+ O\left(q^{-s}\sum_{\mathbf{x}\in\mathbb{F}_q^s} E(\mathbf{x})F(\mathbf{x})q^{-1}\right)\\
&+ O\left(q^{-s}\sum_{\mathbf{x},\mathbf{y}\in\mathbb{F}_q^s:\mathbf{x}\neq\mathbf{y},|\mathbf{x}-\mathbf{y}|^2\neq 0} E(\mathbf{x})F(\mathbf{y})q^{-s/2-1}\right)\\
&= q^{-\frac{3}{2}s}c_q^s\left(\nu(0)+O(\#E)\right)+O\left(q^{-s-1}\sum_{\mathbf{x}\in\mathbb{F}_q^s}E(\mathbf{x})F(\mathbf{x})\right)\\
&+ O\left(q^{-\frac{3}{2}s-1}\sum_{\mathbf{x},\mathbf{y}\in\mathbb{F}_q^s:\mathbf{x}\neq\mathbf{y}} E(\mathbf{x})F(\mathbf{y})\right)\\
&= q^{-\frac{3}{2}s}c_q^s\nu(0)+O\left(q^{-\frac{3}{2}s}\#E\right)\\
&+ O\left(q^{-s-1}\left(\sum_{\mathbf{x}\in\mathbb{F}_q^s}E(\mathbf{x})^2\right)^{1/2}\left(\sum_{\mathbf{x}\in\mathbb{F}_q^s}F(\mathbf{x})^2\right)^{1/2}\right)\\
&+ O\left(q^{-\frac{3}{2}s-1}\sum_{\mathbf{x}\in\mathbb{F}_q^s}E(\mathbf{x})\sum_{\mathbf{y}\in\mathbb{F}_q^s}F(\mathbf{y})\right)\\
&= q^{-\frac{3}{2}s}c_q^s\nu(0)+O\left(q^{-\frac{3}{2}s}\#E\right)+O\left(q^{-s-1}(\#E)^{1/2}(\#F)^{1/2}\right)\\
&+ O\left(q^{-\frac{3}{2}s-1}(\#E)(\#F)\right)\\
&= q^{-\frac{3}{2}s}c_q^s\nu(0)+O\left(q^{-\frac{3}{2}s-1}(\#E)(\#F)\right).
\end{aligned}$$

Multiplying with $\overline{\sigma_{E,F}(0)}$ and noting that $\nu(0)=O\left((\#E)(\#F)\right)$ by Lemma 6 then yields the result. □

Lemma 9. *Let $s\geq 2$, $\#E\leq\#F$ and $(\#E)(\#F)\geq 900q^s$. Then*

$$\sum_{r\in\mathbb{F}_q^*}\nu(r)^2 \ll \frac{(\#E)^2(\#F)^2}{q}+(\log q)q^{\frac{s-1}{2}}(\#E)^2(\#F).$$

For $s=2$, we also have the alternative bound

$$\sum_{r\in\mathbb{F}_q^*}\nu(r)^2 \ll \frac{(\#E)^2(\#F)^2}{q}+O\left((\log q)q(\#E)^{3/2}(\#F)\right).$$

Proof. By Lemma 5, Lemma 7, Lemma 1 and Lemma 8 we obtain

$$\begin{aligned}
\sum_{r\in\mathbb{F}_q} \nu(r)^2 &\leq 4\frac{(\#E)^2(\#F)^2}{q} + \sum_{j\in\mathbb{F}_q} |\delta(j)|^2 \\
&\leq 4\frac{(\#E)^2(\#F)^2}{q} + q^{3s}\,|\sigma_{E,F}(0)|^2 \\
&+ q^{3s} \sum_{r\in\mathbb{F}_q^*} \sigma_E(r)\sigma_F(r) + q^{s-1}(\#E)(\#F) \\
&\leq \nu(0)^2 + 4\frac{(\#E)^2(\#F)^2}{q} + O\left(q^{-1}(\#E)^2(\#F)^2\right) \\
&+ O\left((\log q)\left(q^{s-1}(\#E)(\#F) + q^{\frac{s-1}{2}}(\#E)^2(\#F)\right)\right) \\
&\leq \nu(0)^2 + O\left(\frac{(\#E)^2(\#F)^2}{q}\right) + O\left((\log q) q^{\frac{s-1}{2}}(\#E)^2(\#F)\right).
\end{aligned}$$

Subtracting $\nu(0)^2$ then gives the result. To obtain the alternative bound for $s = 2$, we just use the alternative bound in Lemma 1 and keep the rest of the proof the same. □

5. Proof of Theorem 1

By definition (8) of $\nu(j)$, clearly

$$\sum_{j\in\mathbb{F}_q} \nu(j) = (\#E)(\#F).$$

Hence, by Lemma 6,

$$\left(\sum_{j\in\mathbb{F}_q} \nu(j)\right)^2 - 2\nu(0)^2 \geq \frac{1}{50}(\#E)^2(\#F)^2.$$

Moreover, by Cauchy-Schwarz,

$$\begin{aligned}
\left(\sum_{j\in\mathbb{F}_q} \nu(j)\right)^2 &\leq 2\nu(0)^2 + 2\left(\sum_{j\in\mathbb{F}_q^*} \nu(j)\right)^2 \\
&\leq 2\nu(0)^2 + 2\left(\sum_{j\in\mathbb{F}_q^*} \nu(j)^2\right)\cdot\left(\sum_{j\in\mathbb{F}_q^*:\nu(j)>0} 1\right) \\
&\leq 2\nu(0)^2 + 2\#\Delta(E)\cdot\sum_{j\in\mathbb{F}_q^*} \nu(j)^2.
\end{aligned}$$

Thus

$$\#\Delta(E) \gg \frac{(\#E)^2(\#F)^2}{\sum_{j\in\mathbb{F}_q^*} \nu(j)^2}.$$

The conclusion now follows immediately from Lemma 9.

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